Helically symmetric astrophysical jets

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Astrophysical jets are modeled outside of their accretion disks by the helically symmetric MHD equilibria. The derived equilibria are smooth, have no current sheets, and no discontinuities. The obtained exact solutions to the MHD equilibrium equations generalize the Chandrasekhar equipartition solution and depend upon two arbitrary functions, $\alpha(\psi)$ and the plasma density $\rho(\psi)\geqslant 0$, and $2K+2$ arbitrary parameters $\gamma, a_N, a_{mn}, b_{mn}$. The total kinetic and magnetic energy of the jet and its mass are finite in any layer $c_1<z<c_2$. In view of a rapid decreasing of the magnetic field $\mathbf{B}$ in the transversal direction, the modeled jets are highly collimated.

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I. INTRODUCTION

As is known, optical and radio observations have discovered a variety of jets from the active galactic nuclei and the young stellar objects; see [1,2]. Among those are the jet in the elliptical galaxy Messier 87 [3,4] in the Virgo cluster and the Herbig-Haro 34 jet [5,6]. The optical radiation emitted by the M87 jet is synchrotron [3]; its intensity image appears to have no axial symmetry [4]. We model these highly collimated jets outside of their accretion disks by the exact magnetohydrodynamics (MHD) equilibria, which are invariant with respect to the helical transformations

$$z \to z+\gamma h, \quad \phi \to \phi + h, \quad r \to r. \tag{1.1}$$

Here r, z, ϕ are the cylindrical coordinates, $\gamma=\text{const}$, and h is an arbitrary parameter.

The observations of the solar prominences detect tubes of enhanced plasma density having helical structure, which are presumed to coincide with the magnetic flux ropes [7]. It is known that the solar prominences can remain in a stable equilibrium for several months.

The laboratory experiments indicate the presence of helical external and internal kink modes of plasma in tokamaks [8–10]. The external kink modes are known as the most dangerous MHD instabilities arising in a plasma pinch. The internal ($m=1$) kink instability is associated with the helical plasma distortion that precedes the sawtooth oscillations [8,9] near the magnetic axis of a tokamak. The best available analytical approximations of the external and internal kink modes for a "straight tokamak" are the helically symmetric plasma equilibria. They have been employed in the numerical study of the reversed-field pinch [10].

All these helically symmetric plasma configurations are described by the JFKO equation derived by Johnson, Frieman, Kulsrud, and Oberman in [11]. This equation is a reduction of the plasma equilibrium equations,

$$\mathbf{J}\times\mathbf{B}=\text{grad}\, p, \quad \mathbf{J}=\frac{1}{\mu}\text{curl}\,\mathbf{B}, \quad \text{div}\,\mathbf{B}=\mathbf{0} \tag{1.2}$$

for the helically symmetric solutions. Here $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, and μ is the magnetic permeability. The JFKO equation reduces to the Grad-Shafranov equation [12,13] when the helical symmetry turns into the axial one. Both equations were widely used to model astrophysical plasmas and laboratory plasmas in tokamaks, stellarators, heliotrons, and torsatrons. However, all exact solutions to the JFKO equation and the Grad-Shafranov equation found during the past four decades either have singularities or unboundedly grow at infinity or are not localized [14–17]. Such solutions have a very restricted applicability.

The system of magnetohydrodynamics equilibrium equations has the form

$$\rho\mathbf{V}\times\text{curl}\,\mathbf{V}-\frac{1}{\mu}\mathbf{B}\times\text{curl}\,\mathbf{B}=\text{grad}\,P+\tfrac{1}{2}\,\rho\,\text{grad}\,\mathbf{V}^2, \tag{1.3}$$

$$\text{div}(\rho\mathbf{V})=\mathbf{0}, \quad \text{div}\,\mathbf{B}=\mathbf{0}, \quad \text{curl}(\mathbf{V}\times\mathbf{B})=\mathbf{0}, \tag{1.4}$$

where $\mathbf{B}$ is the magnetic vector field, μ is the constant magnetic permeability, $\mathbf{V}$ is the velocity vector field of the ideal fluid, $\rho=\rho(x)$ is its density, and P is the pressure. For $\mathbf{V}=\mathbf{0}$, Eqs. (1.3) and (1.4) coincide with the plasma equilibrium equations (1.2). For $\mathbf{B}=\mathbf{0}$, Eqs. (1.3) and (1.4) describe equilibria of an ideal fluid.

In this paper, we develop a helically symmetric model of astrophysical jets that are in a state of magnetohydrodynamics equilibrium. Such equilibria have to be global, which means they have to satisfy the following necessary physical conditions in the cylindrical coordinates r, z, ϕ.

(a) The magnetic field $\mathbf{B}$, the plasma velocity $\mathbf{V}$, and pressure P are smooth and bounded in $\mathbb{R}^3$.

(b) The total magnetic and kinetic energy of plasma and its total mass are finite in any layer $c_1<z<c_2$. The pressure $P\to p_1$ at $r\to\infty$.

(c) All magnetic-field lines are bounded in the radial variable r.

The conditions (b) and (c) mean that the jet is localized in the transversal direction. The asymptotic value of pressure p_1 in the condition (b) is the average pressure of the ambient medium. As usual, the gravitational force $-\rho\,\text{grad}\,\Psi$ is included in the pressure gradient in Eq. (1.3) in the approximation of constant density ρ.

We model the astrophysical jets by the helically symmetric MHD equilibria, which are smooth and bounded in the whole Euclidean space $\mathbb{R}^3$ and have no singularities, no current sheets, and no discontinuities. In [18,19], we derived the exact global plasma equilibria with axial symmetry. The obtained magnetohydrodynamics equilibria generalize the Chandrasekhar equipartition solution [20], which is defined by the following three conditions:

$$\mathbf{V}=\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B},\quad \rho=\text{const},\quad P+\frac{1}{2\mu}\mathbf{B}^2=\text{const}. \quad (1.5)$$

The first condition (1.5) implies that the densities of the plasma kinetic energy $\rho\mathbf{V}^2/2$ and magnetic energy $\mathbf{B}^2/(2\mu)$ are equal. However, in the real plasma equilibria, the equipartition condition might not be satisfied [1].

We suppose that

$$\mathbf{B}=a(\psi)\mathbf{B}_1,\quad \mathbf{V}=\frac{b(\psi)}{\sqrt{\mu\rho(\psi)}}\mathbf{B}_1, \quad (1.6)$$

where the vector field $\mathbf{B}_1$ and the functions $a(\psi),b(\psi),\psi=\psi(x)$ satisfy the equations (see Appendix C)

$$a^2(\psi)-b^2(\psi)=C_1=\text{const},\quad (\mathbf{B}_1\cdot\text{grad}\,\psi)=\mathbf{0},\quad \text{div}\,\mathbf{B}_1=\mathbf{0}. \quad (1.7)$$

It is evident that Eqs. (1.4) follow from Eqs. (1.6) and (1.7).

The second equation (1.7) means that the function $\psi(x)$ is constant along the magnetic-field lines, or along the plasma streamlines. Hence the surfaces of the constant level of each function $a(\psi)$, $b(\psi)$, and $\rho(\psi)$ are the magnetic surfaces. The two functions $b(\psi)$ and $\rho(\psi)\geqslant 0$ are arbitrary.

For the special case $b(\psi)=c\rho(\psi)=f^2(\psi)$, in which function $f(\psi)$ is nonzero in an interval I: $0<\psi_1<\psi<\psi_2$ and is zero outside of I, the plasma is localized in a cylindrical domain $0\leqslant r<R_0$ around the axis z and a magnetic field exists in the whole space. It is evident that the total mass of plasma and its kinetic energy are finite in any layer $c_1<z<c_2$.

The solutions (1.6) have the following physical meaning. The ratio of the plasma magnetic and kinetic energy

$$\frac{\mathbf{B}^2}{\mu\rho\mathbf{V}^2}=1+\frac{C_1}{b^2(\psi)}$$

is constant on the magnetic surfaces but is variable in the space $\mathbb{R}^3$. Note that neither magnetic nor kinetic energy separately are constant on the magnetic surfaces. If $C_1>0$, then the magnetic energy is everywhere greater than the kinetic energy; if $C_1<0$, the converse is true. The case $C_1=0$, $\rho=\text{const}$ corresponds to the Chandrasekhar equipartition solution (1.5).

We show that Eqs. (1.3) and (1.4) for solutions (1.6) and (1.7) are reduced to the plasma equilibrium equations (1.2). Hence their helically symmetric solutions are described by the JFKO equation.

For the helically symmetric MHD equilibria, the magnetic surfaces are nested cylinders that rotate as helices around the axis z. The innermost cylinders are their magnetic axes, which altogether form a finite system of helices. The simplest equilibria have a double helix of magnetic axes. These exact solutions can be used to model not only the astrophysical jets but also the plasma ($m\geqslant 1$) kink modes detected in tokamaks [14,16] and the twisted magnetic flux ropes observed in solar prominences [7].

An important feature of the obtained solutions is that their generic magnetic-field lines and streamlines are quasiperiodic in variables r,z,ϕ. That means that they never repeat in the z direction but appear arbitrarily close to their initial data when variable z evolves. Therefore, the pattern of winding of the magnetic-field lines about each other does vary in the z direction.

The intensity of the synchrotron emission from the jet is proportional to $\mathbf{B}^2$ [3]. Hence the jet intensity image is defined by the form and the distribution of the surfaces $\mathbf{B}^2=\text{const}$. For the derived solutions, the magnetic field $\mathbf{B}$ decreases at $r\to\infty$ as rapidly as $c_N\exp(-\beta r^2)r^{2N}$. Hence the high collimation of the jet follows.

II. THE MAIN PHYSICAL ASSUMPTIONS

We develop a model of astrophysical jets based on the following physical assumptions inspired by the results of the observations of the jet in the elliptical galaxy Messier 87 [4].

(i) The jet is in a state of a helically symmetric MHD equilibrium.

(ii) In the cylindrical coordinates r,z,ϕ (1.1), the total plasma kinetic energy and magnetic energy in any layer $c_1<z<c_2$ are finite.

(iii) All plasma streamlines and all magnetic-field lines are bounded in the radial variable r.

(iv) At $r\to\infty$, the plasma pressure $P\to p_1$, where $p_1=\text{const}>0$ is the average pressure of the ambient medium.

(v) The total mass of plasma in any layer $c_1<z<c_2$ is finite.

Let us prove that the MHD equilibrium equations (1.3) and (1.4) for solutions (1.6) and (1.7) are reduced to the plasma equilibrium equations (1.2). For any aligned vector fields $\mathbf{A}=g(x)\mathbf{B}_1$, one has the identity

$$\mathbf{A}\times\text{curl}\,\mathbf{A}=g^2\mathbf{B}_1\times\text{curl}\,\mathbf{B}_1+\mathbf{B}_1^2\,\text{grad}\,g^2/2 -g(\mathbf{B}_1\cdot\text{grad}\,g)\mathbf{B}_1. \quad (2.1)$$

Applying identity (2.1) to the vector fields (1.6) and using Eqs. (1.7), we get

$$\mathbf{B}\times\text{curl}\,\mathbf{B}=a^2\mathbf{B}_1\times\text{curl}\,\mathbf{B}_1+\tfrac{1}{2}\mathbf{B}_1^2\,\text{grad}\,a^2,$$

$$\rho\mathbf{V}\times\text{curl}\,\mathbf{V}=\frac{1}{\mu}b^2\mathbf{B}_1\times\text{curl}\,\mathbf{B}_1+\frac{1}{2\mu}\rho\mathbf{B}_1^2\,\text{grad}\,\frac{b^2}{\rho}.$$

Substituting these formulas into the MHD equation (1.3), we obtain

$$\frac{1}{\mu}(a^2-b^2)\text{curl}\,\mathbf{B}_1\times\mathbf{B}_1 =\text{grad}\,P+\frac{1}{2\mu}(\mathbf{B}_1^2\,\text{grad}\,a^2+b^2\,\text{grad}\,\mathbf{B}_1^2).$$

Using here the equation $a^2(\psi)-b^2(\psi)=C_1$, we arrive at the plasma equilibrium equations

$$\operatorname{curl}\mathbf{B}_1\times\mathbf{B}_1=\mu\operatorname{grad}p,\quad \operatorname{div}\mathbf{B}_1=\mathbf{0}, \tag{2.2}$$

where

$$p=\left(P+\frac{1}{2\mu}b^2(\psi)\mathbf{B}_1^2\right)\Big/ C_1. \tag{2.3}$$

Remark 1. The plasma equilibrium equations (2.2) imply the equation [21] $(\mathbf{B}_1\cdot\operatorname{grad}p)=0$. Hence for any solution to Eqs. (2.2), function $\psi(x)$ does exist and $\psi(x)=p(x)$. The above arguments prove also that any plasma equilibrium $\mathbf{B}_1(x),p(x)$ defines an infinite dimensional family of magnetohydrodynamics equilibria,

$$\mathbf{B}=a(p)\mathbf{B}_1,\quad \mathbf{V}=\frac{b(p)}{\sqrt{\mu\rho(p)}}\mathbf{B}_1,\quad a^2(p)-b^2(p)=C_1,$$

$$P=C_1p-\frac{b^2(p)}{2\mu}\mathbf{B}_1^2,$$

where $b(p)$ and the plasma density $\rho(p)\geqslant 0$ are arbitrary smooth functions.

The helically symmetric magnetic field $\mathbf{B}_1$ has the form

$$\mathbf{B}_1=\frac{\psi_u}{r}\hat{\mathbf{e}}_r-\frac{\psi_r}{r}\hat{\mathbf{e}}_u+\frac{f}{r}\hat{\mathbf{e}}_\phi, \tag{2.4}$$

where $\psi(r,u)$ and $f(r,u)$ are some smooth functions, $u=z-\gamma\phi$, $\psi_x=\partial\psi/\partial x$, and $\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_u,\hat{\mathbf{e}}_\phi$ are the unit tangent vectors corresponding to the coordinates r,u,ϕ. The current $\mathbf{J}$ is

$$\mathbf{J}=\frac{1}{\mu}\operatorname{curl}\mathbf{B}_1=\frac{1}{\mu}\left(\frac{I_r}{r}\hat{\mathbf{e}}_u-\frac{I_u}{r}\hat{\mathbf{e}}_r+\Phi\hat{\mathbf{e}}_\phi\right), \tag{2.5}$$

where

$$I=\frac{r^2+\gamma^2}{r^2}f-\frac{\gamma}{r}\psi_r, \tag{2.6}$$

$$\Phi=\frac{1}{r}\frac{\partial^2\psi}{\partial u^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)-\gamma\frac{\partial}{\partial r}\left(\frac{f}{r^2}\right). \tag{2.7}$$

Formulas (2.4) and (2.6) imply the expression for the magnetic field $\mathbf{B}_1$ in the cylindrical coordinates r,z,ϕ:

$$\mathbf{B}_1=\frac{\psi_u}{r}\hat{\mathbf{e}}_r+B_1\hat{\mathbf{e}}_z+B_2\hat{\mathbf{e}}_\phi,\quad B_1=\frac{\gamma I(\psi)-r\psi_r}{r^2+\gamma^2},$$

$$B_2=\frac{rI(\psi)+\gamma\psi_r}{r^2+\gamma^2}. \tag{2.8}$$

Formulas (2.4) and (2.8) imply that Eqs. (1.7) are satisfied if and only if $a=a(\psi)$, $b=b(\psi)$, and $\rho=\rho(\psi)$ are arbitrary functions of ψ, $\rho(\psi)\geqslant 0$.

Johnson *et al.* had shown in [11] that the plasma equilibrium equations (2.2) for the helically symmetric solutions are equivalent to the equalities $I=I(\psi)$ and $p=p(\psi)$ with arbitrary functions $I(\psi)$ and $p(\psi)$, and the equation

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial u^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\frac{\partial\psi}{\partial r}\right)+\frac{II'(\psi)}{r^2+\gamma^2}+\frac{2\gamma I(\psi)}{(r^2+\gamma^2)^2}=-\mu p'(\psi), \tag{2.9}$$

where the prime denotes differentiation with respect to ψ. Equation (2.9) is equivalent to the Johnson-Frieman-Kulsrud-Oberman (JFKO) equation [11], in which another variable $u=m\phi-hz$ was employed. We use the variable $u=z-\gamma\phi$ because only parameter $\gamma=m/h$ is essential. For $\gamma=0$, the JFKO equation (2.9) turns into the Grad-Shafranov equation [12,13], which describes the axially symmetric plasma equilibria.

The JFKO equation (2.9) and formula (2.6) imply that function Φ (2.7) has the form $\Phi=-\mu rp'(\psi)-fI'(\psi)/r$. Hence we obtain the following for the current $\mathbf{J}$ (2.5) in the cylindrical coordinates r,z,ϕ:

$$\mathbf{J}=-\frac{I'(\psi)}{\mu}\mathbf{B}_1-p'(\psi)(r\hat{\mathbf{e}}_\phi+\gamma\hat{\mathbf{e}}_z). \tag{2.10}$$

For the analytical case, the functions $I(\psi)$ and $p(\psi)$ can be expanded into the power series

$$I(\psi)=\alpha_0+\alpha_1\psi+\alpha_2\psi^2+\cdots,$$

$$p(\psi)=p_0+\beta_1\psi+\beta_2\psi^2+\cdots, \tag{2.11}$$

with constant coefficients α_i,β_j. The physical condition (ii) [the finiteness of the magnetic energy $\mathbf{B}^2/(2\mu)$ in any layer $c_1<z<c_2$] and the formulas (2.8) imply that at $r\to\infty$ the asymptotics hold:

$$I(\psi)\to 0,\quad \psi_r\to 0,\quad \psi_u\to 0. \tag{2.12}$$

The flux function $\psi(r,u)$ is defined by Eq. (2.4) up to an arbitrary constant. Using asymptotics (2.12), we normalize this constant by the condition $\psi(r,u)\to 0$ at $r\to\infty$.

The physical condition (v) implies $\rho(r,z)\to 0$ at $r\to\infty$. Hence the condition $\rho\geqslant 0$ yields

$$\rho=a_2\psi^2+a_3\psi^3+\cdots, \tag{2.13}$$

where $a_2>0$. If $b(\psi)=C_0[\rho(\psi)]^{n/2}$, $n>1$, then the plasma velocity $\mathbf{V}=C_2\rho^{(n-1)/2}\mathbf{B}_1$ is finite everywhere.

The asymptotics (2.12) and $\psi\to 0$ at $r\to\infty$ imply that $\alpha_0=0$. The formula (2.10) and the asymptotics $\mathbf{B}\to\mathbf{0}$, $\mathbf{J}\to\mathbf{0}$ at $r\to\infty$ imply $\beta_1=0$. Hence for any global solutions, the power series (2.11) actually have the form ($\beta_2=\pm\kappa^2$)

$$I(\psi)=\alpha\psi+\alpha_2\psi^2+\cdots,\quad p(\psi)=p_0\pm\kappa^2\psi^2+\cdots. \tag{2.14}$$

Thus, the general global solutions are perturbations of the ground-state solutions defined by the lowest-order terms,

$$I(\psi)=\alpha\psi,\quad p(\psi)=p_0\pm\kappa^2\psi^2.$$

In [24–26], the formula for the pressure $p(\psi)=p_0+\kappa^2\psi^2$ was used to study the Grad-Shafranov equation (axial symmetry), which is the limiting case of the JFKO equation (2.9) at $\gamma\to 0$. It was shown that the corresponding exact solutions

$\psi(r,z)$ have infinitely many zeros at $r\to\infty$ and do not satisfy the physical condition (ii). Therefore, we do not consider this alternative for the pressure.

Thus the physical conditions (i) and (ii) lead us uniquely to the following ground-state assumptions:

$$I(\psi)=\alpha\psi, \quad p(\psi)=p_0-2\beta^2\psi^2/\mu,$$

where α and β are arbitrary constants. The corresponding JFKO equation (2.9) is linear:

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial u^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2+\gamma^2}\frac{\partial\psi}{\partial r}\right)=4\beta^2\psi-\frac{\alpha^2\psi}{r^2+\gamma^2}-\frac{2\alpha\gamma\psi}{(r^2+\gamma^2)^2}. \tag{2.15}$$

In Sec. III, we prove that Eq. (2.15) has exact global solutions that satisfy all physical conditions (i)–(v).

The plasma pressure P (2.3) takes the form

$$P=p_1-\frac{1}{\mu}2\beta^2C_1\psi^2-\frac{1}{2\mu}b^2(\psi)\mathbf{B}_1^2, \tag{2.16}$$

where $p_1=C_1p_0$. We suppose that

$$p_1>\max[2\beta^2C_1\psi^2+b^2(\psi)\mathbf{B}_1^2/2]/\mu. \tag{2.17}$$

Then the plasma pressure P is >0 everywhere. It is evident that constant p_1 does not enter the plasma equilibrium equations (2.2).

Remark 2. For any solution $\psi(r,u)$ to the linear equation (2.15), the function $C\psi(r,u)$ is also a solution for any constant C. Hence, for any bounded solution $\psi(r,u)$ and any ambient pressure $p_1>0$, we have a family of solutions $C\psi(r,u)$ that satisfy the condition (2.16). The physical condition (iv) follows from Eq. (2.16) and the asymptotics $b^2(\psi)\mathbf{B}_1^2=\mu\rho(\psi)\mathbf{V}^2\to 0$, $\psi\to 0$ at $r\to\infty$.

Remark 3. $a^2(\psi)-b^2(\psi)=C_1$ [Eq. (1.7)] has the following solutions. (i) $C_1=k^2$: $a(\psi)=k\,\mathrm{ch}\,\alpha(\psi)$, $b(\psi)=k\,\mathrm{sh}\,\alpha(\psi)$; (ii) $C_1=-k^2$: $a(\psi)=k\,\mathrm{sh}\,\beta(\psi), b(\psi)=k\,\mathrm{ch}\,\beta(\psi)$, where $\alpha(\psi)$ and $\beta(\psi)$ are arbitrary smooth functions.

The corresponding vector fields (1.6) have the form

$$\mathbf{B}=k\,\mathrm{ch}\,\alpha(\psi)\mathbf{B}_1, \quad \mathbf{V}=\frac{k\,\mathrm{sh}\,\alpha(\psi)}{\sqrt{\mu\rho(\psi)}}\mathbf{B}_1,$$

$$\mathbf{B}=k\,\mathrm{sh}\,\beta(\psi)\mathbf{B}_1, \quad \mathbf{V}=\frac{k\,\mathrm{ch}\,\beta(\psi)}{\sqrt{\mu\rho(\psi)}}\mathbf{B}_1.$$

For the first case, the magnetic energy is everywhere greater than the kinetic energy; the converse is true for the second case.

III. EXACT GLOBAL MHD EQUILIBRIA

The helically symmetric solutions depend on the two variables $u=z-\gamma\phi$ and r (1.1) and therefore have to be periodic in the variable u (and hence in the variable z) with period $2\pi\gamma$. To study the linear equation (2.15), we apply the Fourier method and separate variables by the substitution

$$\psi(r,u)=A(r)[a\cos(\omega u)+b\sin(\omega u)], \quad u=z-\gamma\phi. \tag{3.1}$$

Hence we obtain the equation

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{r^2+\gamma^2}\frac{dA(r)}{dr}\right)=\left(4\beta^2+\frac{\omega^2}{r^2}-\frac{\alpha^2}{r^2+\gamma^2}-\frac{2\alpha\gamma}{(r^2+\gamma^2)^2}\right)A(r). \tag{3.2}$$

This equation does not belong to any known class of integrable ones [22]. To find exact solutions, we substitute $A(r)=r^{\lambda}e^{-\beta r^2}B(x)$, $x=2\beta r^2$, $\lambda\geqslant 0$, which reduces Eq. (3.2) to

$$(x^2+c_1x)B''+[-x^2+(\lambda-c_1)x+(\lambda+1)c_1]B'+\left(\frac{\alpha^2-\eta}{8\beta}x+\frac{\alpha^2\gamma^2-\eta\gamma^2}{4}+\frac{\alpha\gamma-c_1-\lambda}{2}+\frac{x+c_1}{4x}\times(\lambda^2-\gamma^2\omega^2)\right)B=0, \tag{3.3}$$

where $B'=dB(x)/dx$, $c_1=2\beta\gamma^2$, and $\eta=4\beta^2\gamma^2+4\beta\lambda+\omega^2$. To find solutions $\psi(r,u)$ (3.1) obeying the asymptotics $\psi\to 0$ at $r\to\infty$, we seek the polynomial functions $B(x)$ (with a nonzero free term). Inspecting the highest- and the lowest-order terms in Eq. (3.3), we obtain the necessary conditions

$$\frac{\alpha^2-\eta}{8\beta}=n, \quad \lambda=|\gamma\omega|, \tag{3.4}$$

where the integer $n\geqslant 0$ is the order of the polynomial $B(x)$. The form of the solution (3.1) implies that $\gamma\omega$ must be an integer: $|\gamma\omega|=m\geqslant 0$. Hence we get $\lambda=m$, $\omega=\pm m/\gamma$, and $\eta=(2\beta\gamma+m/\gamma)^2$. The first necessary condition (3.4) becomes the algebraic equation

$$\alpha^2\gamma^2=(m+c_1)^2+4nc_1 \tag{3.5}$$

for the two unknown integers m and n. We present Eq. (3.3) in the form

$$(x^2+c_1x)B''+[-x^2+(m-c_1)x+(m+1)c_1]\times B'+n(x+c_1-k_{mn}c_1)B=0, \tag{3.6}$$

where $k_{mn}=(m+c_1-\alpha\gamma)/(2nc_1)$. Note that Eq. (3.6) is different from all classical differential equations that define the Chebyshev, Hermite, Laguerre, Legendre, or Jacobi polynomials [22,23].

In Appendix A, we prove that the differential equation (3.6) has a polynomial solution

$$B_{mn}(x)=\frac{d^m}{dx^m}L_{m+n}(x)-k_{mn}x\frac{d^{m+1}}{dx^{m+1}}L_{m+n}(x), \tag{3.7}$$

where $L_p(x)$ are the Laguerre polynomials

$$L_p(x)=\frac{1}{p!}e^x\frac{d^p}{dx^p}(e^{-x}x^p)=\sum_{k=0}^{p}\frac{(-1)^k p!}{(k!)^2(p-k)!}x^k. \quad (3.8)$$

Hence we obtain that Eq. (2.15) has the exact solution

$$\psi_{mn}=r^m e^{-\beta r^2}B_{mn}(2\beta r^2)[a_{mn}\cos(mu/\gamma)+b_{mn}\sin(mu/\gamma)], \quad (3.9)$$

where a_{mn} and b_{mn} are arbitrary constants. For $m\geqslant 2$, the magnetic field (2.8) $\mathbf{B}_1(0,z)=0$ on the axis $r=0$. Only the $m=1$ kink mode (3.9) has a nonzero magnetic field at $r=0$.

The linearity of Eq. (2.15) implies that if the algebraic equation (3.5) has several integral solutions m, n, then any linear combination of the corresponding functions (3.9) is an exact solution to Eq. (2.15).

For $n=0$, the necessary condition (3.5) is $\alpha=2\beta\gamma+m/\gamma$, and Eq. (3.6) has solution $B=\text{const}$. Hence the JFKO equation (2.9) has the exact solution

$$\psi_{m0}(r,u)=r^m e^{-\beta r^2}[a_m\cos(mu/\gamma)+b_m\sin(mu/\gamma)]. \quad (3.10)$$

For $m=n=0$, $\alpha=2\beta\gamma$, the solution (3.10) takes the Gaussian function form $\psi_0(r)=\exp(-\beta r^2)$. The corresponding magnetic field (2.8), current (2.10), and pressure p are

$$\mathbf{B}_1=2\beta e^{-\beta r^2}\hat{\mathbf{e}}_z,\quad \mathbf{J}=\frac{4\beta^2}{\mu}re^{-\beta r^2}\hat{\mathbf{e}}_\phi,$$
$$p=p_0-\frac{2\beta^2}{\mu}e^{-2\beta r^2}. \quad (3.11)$$

This is a global plasma equilibrium possessing cylindrical symmetry.

For $m=0$ and $n=N\neq 0$, the necessary condition (3.5) is

$$\alpha^2\gamma^2=c_1^2+4Nc_1 \quad (3.12)$$

and the exact solution (3.9) takes the form

$$\psi_N(r)=a_N e^{-\beta r^2}B_{0N}(2\beta r^2), \quad (3.13)$$

where polynomials $B_{0N}(x)$ are

$$B_{0N}(x)=L_N(x)-k_N x L_N'(x). \quad (3.14)$$

Equation (3.12) implies

$$\alpha\gamma=c_1\sqrt{1+2N/(\beta\gamma^2)},$$
$$k_N=\frac{c_1-\alpha\gamma}{2Nc_1}=\frac{1-\sqrt{1+2N/(\beta\gamma^2)}}{2N}<0. \quad (3.15)$$

Hence polynomials $B_{0N}(x)$ depend on parameter $\beta\gamma^2$.

For the flux function (3.13), the magnetic field (2.8)

$$\mathbf{B}_1=\frac{\alpha\gamma\psi_N-r\psi_N'}{r^2+\gamma^2}\hat{\mathbf{e}}_z+\frac{\alpha r\psi_N+\gamma\psi_N'}{r^2+\gamma^2}\hat{\mathbf{e}}_\phi \quad (3.16)$$

decreases at $r\to\infty$ as rapidly as $c_N\exp(-\beta r^2)r^{2N}$. Hence the total magnetic energy in any layer $z_1<z<z_2$ is finite. The magnetic surfaces $\psi_N(r)=\text{const}$ are cylinders $r=\text{const}$.

Thus we have demonstrated that the JFKO equation (2.9) has global solutions (3.11), (3.13), (3.16), where the two parameters $\beta>0$ and $\gamma\neq 0$ are arbitrary. These equilibria possess cylindrical symmetry because they are z- and ϕ-invariant.

For $m\geqslant 1$, the flux functions $\psi_{mn}(r,u)$ (3.9) define the magnetic fields $\mathbf{B}_1$ (2.8) and currents $\mathbf{J}$ (2.10), which are smooth in the Cartesian coordinates and decrease at $r\to\infty$ as rapidly as $c_n\exp(-\beta r^2)r^{2n}$. Hence the above physical condition (ii) is satisfied. However, condition (iii) is not met because the magnetic surface $\psi_{mn}(r,u)=0$ is a helicoid $z=\gamma\phi+c$ that is unbounded in variable r.

To find the global equilibria, we consider a linear combination of the exact solutions $\psi_N(r)$ (3.13) and $\psi_{mn}(r,u)$ (3.9) that satisfies Eq. (2.15) provided that the two necessary conditions (3.5) and (3.12) hold simultaneously. These two conditions yield the formulas

$$c_1=2\beta\gamma^2=\frac{m^2}{2(2N-2n-m)},$$
$$\alpha\gamma=\frac{m\sqrt{(4N-m)^2-16nN}}{2(2N-2n-m)}, \quad (3.17)$$

$$k_{mn}=\frac{m+c_1-\alpha\gamma}{2nc_1}=\frac{1}{2mn}[4(N-n)-m-\sqrt{(4N-m)^2-16nN}]. \quad (3.18)$$

Thus we obtain that for any value of parameter γ and $\beta=\beta(\gamma)$ defined by Eqs. (3.17), the JFKO equation (2.9) has the exact solutions

$$\psi_{Nmn}=e^{-\beta r^2}\{a_N B_{0N}(x)+r^m B_{mn}(x)[a_{mn}\cos(mu/\gamma)+b_{mn}\sin(mu/\gamma)]\}, \quad (3.19)$$

where N,m,n are arbitrary integers $\geqslant 0$ satisfying the inequality $2N>2n+m$, and $x=2\beta r^2$. The inequality $2N>2n+m$ implies that function $\psi_{Nmn}(r,u)$ (3.19) has the leading term $(-2\beta)^N a_N\exp(-\beta r^2)r^{2N}$ at $r\to\infty$. Hence equilibria (3.19) for $m\geqslant 1$ satisfy the above condition (ii), and all magnetic surfaces $\psi_{Nmn}(r,u)=\text{const}$ asymptotically for $r\gg 1$ are cylinders $r=\text{const}$. Therefore, all streamlines and all magnetic-field lines are bounded in variable r, condition (iii). The formula (3.19) implies that the total mass of plasma in any layer $c_1<z<c_2$ is finite if, for example, $\rho(\psi)=a\psi^2$ or $\rho(\psi)$ is any analytical function of ψ (2.13). Thus the exact solutions (3.19) define the global MHD equilibria satisfying the above physical conditions (i)–(v).

IV. EXAMPLES

For $N=1$, the inequality $2N>2n+m$ has one integral solution $m=1$, $n=0$. Formulas (3.17) give $c_1=\frac{1}{2}$, $\alpha\gamma=\frac{3}{2}$, and $\beta=1/(4\gamma^2)$; hence $k_1=-1$ and polynomial $B_{01}(x)$ is

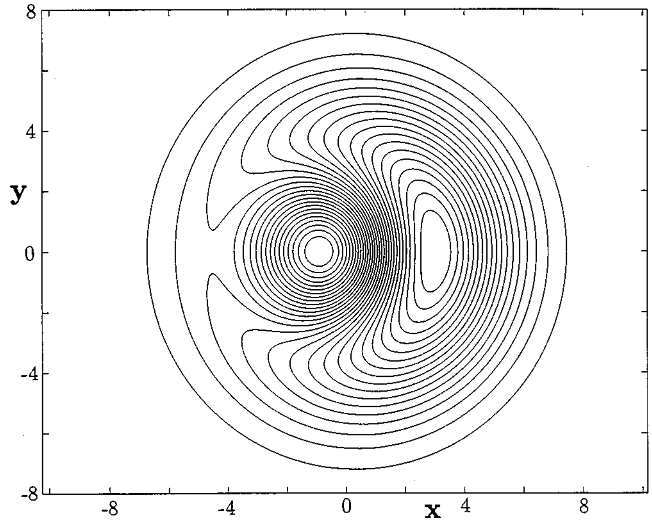

FIG. 1. Magnetic surfaces at $z=0$ for the astrophysical jets model.

$1-2x$; see formula (A2) in Appendix A below. Thus the flux function (3.19) has the equivalent form ($x=2\beta r^2$)

$$\psi(r,z,\phi)=e^{-\beta r^2}[1-4\beta r^2+a_1 r\cos(z/\gamma-\phi)]. \quad (4.1)$$

Figure 1 shows the section $z=0$ of the magnetic surfaces $\psi(r,z,\phi)=\text{const}$ for $a_1=-1$, $\beta=0.1$, and $\gamma=\sqrt{\frac{5}{2}}$. The function $\psi(r,0,\phi)$ (4.1) achieves its maximum at $r=0.8968$, $\phi=\pi$ and its minimum at $r=3.0168$, $\phi=0$.

The two-dimensional magnetic surfaces are obtained from the curves in Fig. 1 by simultaneous rotation in ϕ with angular speed 1 and translation in z with speed γ. Figure 1 shows that there are three domains filled with cylindrical magnetic surfaces: one outer domain and two inner domains with two magnetic axes corresponding to the critical points of maxima and minima. The magnetic axes form a *double helix* that is presented in Fig. 2. These two curves are exact magnetic-field lines.

Figure 3 represents the magnetic energy density $\mathbf{B}_1^2(x,y,z)/2\mu$ for the plasma equilibrium (4.1) for $y=0$, $z=0$ and $a_1=-1$, $\beta=0.1$, $\gamma=\sqrt{\frac{5}{2}}$, $\mu=0.1$. Figure 4 shows the level curves $\mathbf{B}_1^2(x,y,0)=\text{const}$, $z=0$. The magnetic energy $\mathbf{B}_1^2/2\mu$ is concentrated near the axis z, and tends to zero

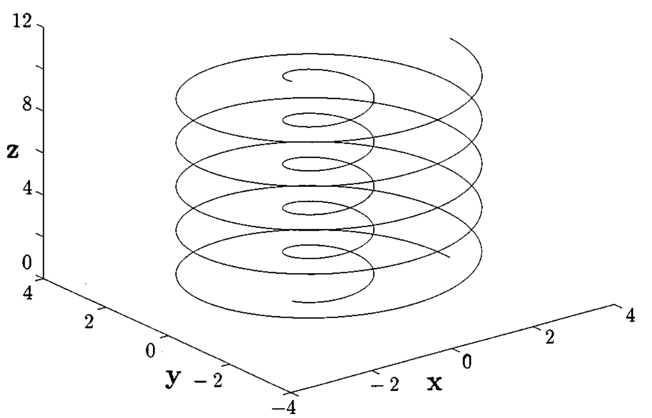

FIG. 2. Double helix of magnetic axes.

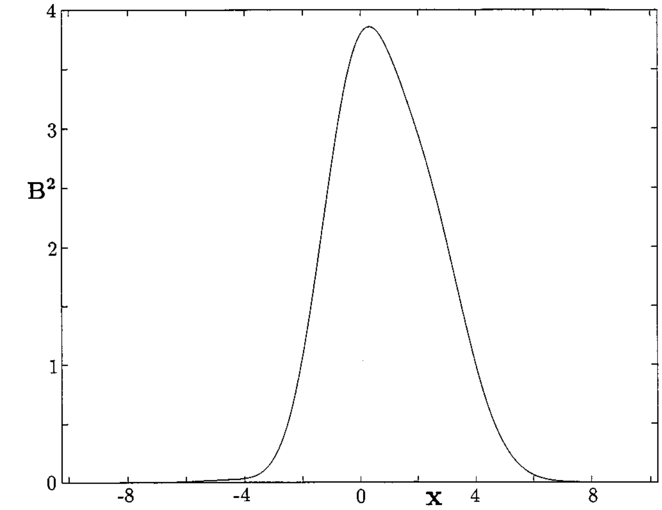

FIG. 3. Density of magnetic energy $\mathbf{B}^2(x,0,0)/2\mu$.

very rapidly at $r\to\infty$. This property means that the above exact solution models a highly collimated astrophysical jet. The intensity image of the jet is defined by its synchrotron radiation, which is proportional to $\mathbf{B}_1^2$ [3]. Hence the detectable form of the jet approximately coincides with a surface $\mathbf{B}_1^2(x,y,z)=\text{const}$. The surfaces $\mathbf{B}_1^2=\text{const}$ are obtained from the curves in Fig. 4 by the helical transformations. It is evident from Figs. 1 and 4 that the magnetic surfaces $\psi=\text{const}$ [or $\rho(\psi)=\text{const}$] and surfaces $\mathbf{B}_1^2=\text{const}$ are different. The magnetic energy is $\mathbf{B}^2/2\mu=a^2(\psi)\mathbf{B}_1^2/2\mu$. Hence the surfaces of constant magnetic energy $\mathbf{B}^2/2\mu$ do not coincide with the magnetic surfaces $\psi=\text{const}$ and depend upon an arbitrary function $a(\psi)$.

Figure 5 shows the plasma pressure $p(x,y,z)=p_0-2\beta^2\psi^2/\mu$ at $y=z=0$, $p_0=1.2$ for the plasma equilibrium (4.1). The plots in Figs. 3 and 5 evidently have no symmetry with respect to $x=0$.

For $N=2$, the inequality $2N>2n+m$ has four integral solutions $m=1$, $n=0$; $m=1$, $n=1$; $m=2$, $n=0$; and $m=3$, $n=0$. For $m=1$, $n=0$, we find from formulas (3.17) $c_1=\frac{1}{6}$, $\alpha\gamma=\frac{7}{6}$, and $\beta=1/(12\gamma^2)$. Hence $k_2=-\frac{3}{2}$ and poly-

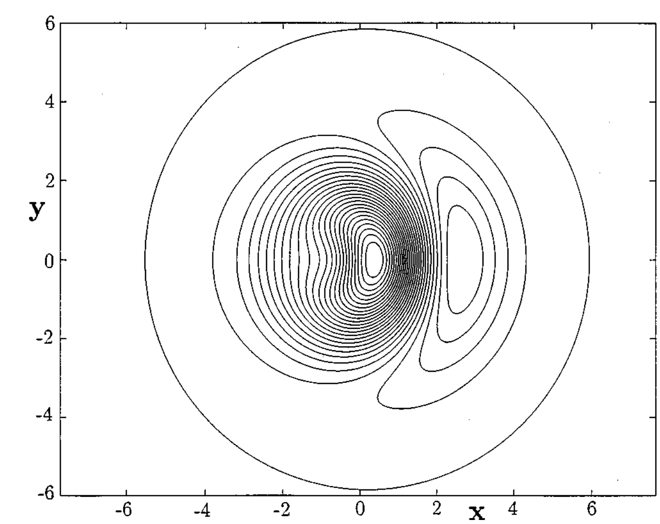

FIG. 4. Surfaces of constant magnetic energy $\mathbf{B}^2(x,y,z)=\text{const}$ at $z=0$.

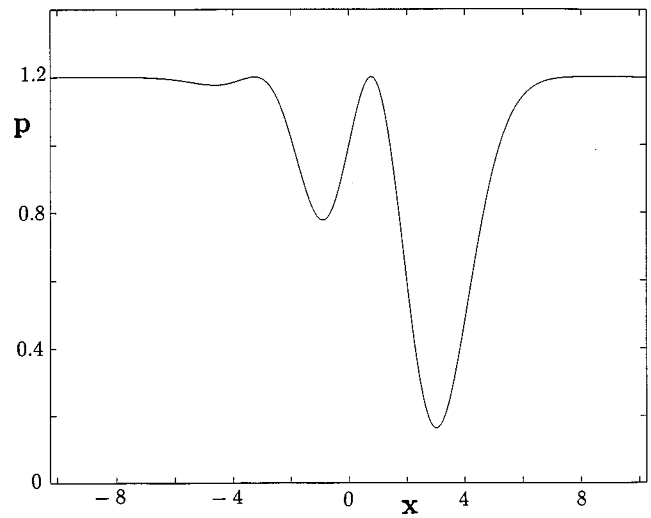

FIG. 5. Plasma pressure $p(x,0,0)$ for the astrophysical jets model.

nomial $B_{02}(x)$ is $1-5x+2x^2$; see Appendix A, formula (A2). The exact solution (3.19) takes the equivalent form

$$\psi(r,z,\phi)=e^{-\beta r^2}[1-10\beta r^2+8\beta^2 r^4+a_1 r\cos(z/\gamma-\phi)]. \tag{4.2}$$

The section $z=0$ of the magnetic surfaces $\psi(r,z,\phi)=\text{const}$ is shown in Fig. 6 for $a_1=-1.5$, $\beta=0.1$, and $\gamma=\sqrt{\frac{5}{6}}$. The flux function $\psi(r,0,\phi)$ (4.2) achieves its local maxima at two points $r=4.8625$, $\phi=\pi$ and $r=0.6940$, $\phi=\pi$ and has its minimum at $r=2.2361$, $\phi=0$. Hence there are four invariant domains in $\mathbb{R}^3$ filled with cylindrical magnetic surfaces: one outer domain and three inner domains. The corresponding three magnetic axes form a *triple helix* that is presented in Fig. 7. The curves in Figs. 1 and 6 are symmetric with respect to the reflection $y\rightarrow -y$.

In the coordinates r,z,ϕ, the magnetic surfaces $\psi(r,u)=h=\text{const}$ are the helically rotating cylinders C_h, see Fig. 1. The sections S_k, $z=2\pi\gamma k$ of a cylinder C_h, are the same for all integers k because of the periodicity $z\rightarrow z+2\pi\gamma$ of

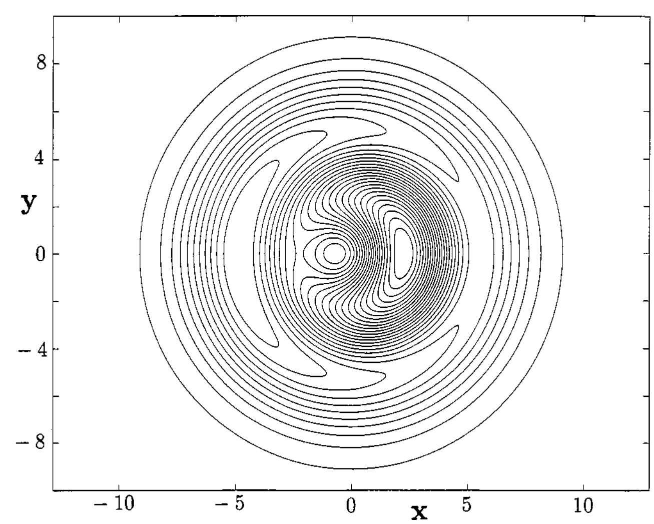

FIG. 6. Magnetic surfaces for plasma equilibrium (4.2) at $z=0$.

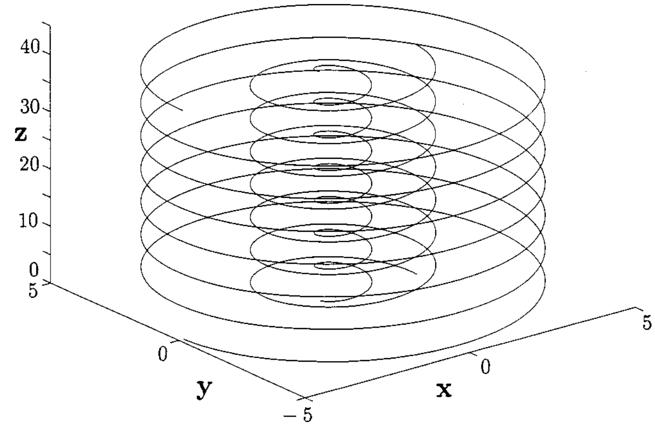

FIG. 7. Triple helix of magnetic axes.

the flux functions (3.19). Hence the subsequent intersections of the magnetic-field lines on C_h with curves S_0 and S_1 define a rotation transform T_h: $S_0\rightarrow S_0=S_1$. For an appropriate angular parameter $s=s \bmod 1$ on the closed curve S_0, the mapping T_h is $T_h(s)=s+\theta_0(h)$. For a generic h, the rotation number $\theta_0(h)$ is irrational. Hence the generic magnetic-field lines are quasiperiodic in variables r, ϕ and in variable $z \bmod 2\pi\gamma$. This important property implies that the pattern of winding of the magnetic-field lines about each other *does vary* in the z direction for the constructed global plasma equilibria (3.19). For details, see Appendix B.

V. HELICAL MHD EQUILIBRIA DEPENDING ON 2K+2 PARAMETERS

To derive the MHD equilibria depending on more than three parameters, we have to find more than two integral solutions to the algebraic equations (3.5) and (3.12) with given constants $\alpha\gamma$ and c_1. The first formula (3.17) implies that parameter $c_1=2\beta\gamma^2$ should be rational, $c_1=p/q$. Excluding $\alpha\gamma=\sqrt{c_1^2+4Nc_1}$ from Eqs. (3.5) and (3.12), we obtain the Diophantine equation [27]

$$4pN=qm^2+2pm+4pn. \tag{5.1}$$

In general, Eq. (5.1) for given N,p,q has many integral solutions m,n. Taking a linear combination of the corresponding flux functions $\psi_N(r)$ (3.13) and $\psi_{mn}(r,u)$ (3.9), we obtain an exact solution to the linear equation (2.15). We present this construction in an explicit form for $c_1=1/(2l)$, where l is an arbitrary odd number. Hence $\beta=c_1/(2\gamma^2)=1/(4l\gamma^2)$ and $\alpha=\sqrt{8lN+1}/(2l\gamma)$.

For $p=1$, $q=2l$, Eq. (5.1) takes the form

$$2N=lm^2+m+2n. \tag{5.2}$$

It is evident that this Diophantine equation has $K+1$ pairs of integral solutions,

$$n=N-\frac{m(lm+1)}{2},\quad m=0,1,2,...,K,\quad K=\left[\frac{\sqrt{8lN+1}-1}{2l}\right].$$

The coefficients k_N (3.15) and k_{mn} (3.18) are

$$k_N = \frac{1-\sqrt{8lN+1}}{2N}, \quad k_{mn} = \frac{2lm+1-\sqrt{8lN+1}}{2n}. \quad (5.3)$$

Taking a linear combination of the exact solutions $\psi_N(r)$ (3.13) and $\psi_{mn}(r,u)$ (3.9) for $1 \leqslant m \leqslant K$, we obtain that the flux function

$$\psi = e^{-\beta r^2}\left(a_N B_{0N}(x) + \sum_{m=1}^{K} r^m B_{mn}(x)[a_{mn}\cos(mu/\gamma) + b_{mn}\sin(mu/\gamma)] \right) \quad (5.4)$$

satisfies the JFKO equation (2.9), depends on an arbitrary parameter γ, and linearly depends upon $2K+1$ arbitrary parameters a_N, a_{mn}, b_{mn}. The corresponding magnetic field $\mathbf{B}$ (2.8), current $\mathbf{J}$ (2.10), and pressure $p = p_0 - 2\beta^2\psi^2/\mu$ are smooth and bounded in the whole Euclidean space. The function $\psi(r,u)$ (5.4) has the leading term $(-2\beta)^N a_N \exp(-\beta r^2) r^{2N}$ at $r \to \infty$. Hence asymptotically for $r \gg 1$, all magnetic surfaces $\psi(r,u) = \text{const}$ are cylinders $r = \text{const}$ and the magnetic field $\mathbf{B}_1$ and current $\mathbf{J}$ decrease at $r \to \infty$ as rapidly as $c_N \exp(-\beta r^2) r^{2N}$ and $p \to p_0$. Therefore, the flux functions $\psi(r,u)$ (5.4) define global plasma equilibria satisfying the above physical conditions (i)–(v).

We apply the previous constructions for $m=1$, $\beta = 1/(4\gamma^2)$. For $N=3$, the Diophantine equation (5.2) has three pairs of integral solutions ($m=0$, $n=3$), ($m=1$, $n=2$), and ($m=2$, $n=0$). Formulas (5.3) imply $k_3 = -\frac{2}{3}$, $k_{12} = -\frac{1}{2}$. Hence polynomial $B_{03}(x)$ is $1-5x+7x^2/2-x^3/2$ and polynomial $B_{12}(x)$ is $-3+9x/2-x^2$; see formulas (A2) and (A3) in Appendix A below. The flux function (5.4) takes the equivalent form

$$\psi = e^{-\beta r^2}\{1 - 10\beta r^2 + 14\beta^2 r^4 - 4\beta^3 r^6 + a_1 r(-3 + 9\beta r^2 - 4\beta^2 r^4)\cos(z/\gamma - \phi) + r^2[a_2\cos(2z/\gamma - 2\phi) + b_2\sin(2z/\gamma - 2\phi)]\}. \quad (5.5)$$

Figure 8 represents the section $z=0$ of the magnetic surfaces $\psi(r,z,\phi) = \text{const}$ for $a_1 = \frac{2}{5}$, $a_2 = -\frac{1}{15}$, $b_2 = \frac{1}{30}$, $\beta = 0.1$, and $\gamma = \sqrt{\frac{5}{2}}$. The function $\psi(r,0,\phi)$ (5.5) has two points of local maxima and three points of local minima that define five helical magnetic axes. The distribution of curves in Fig. 8 is evidently asymmetric while the curves in Figs. 1 and 6 are symmetric with respect to the reflection $y \to -y$. The generic plasma equilibria (5.4) have no additional symmetries.

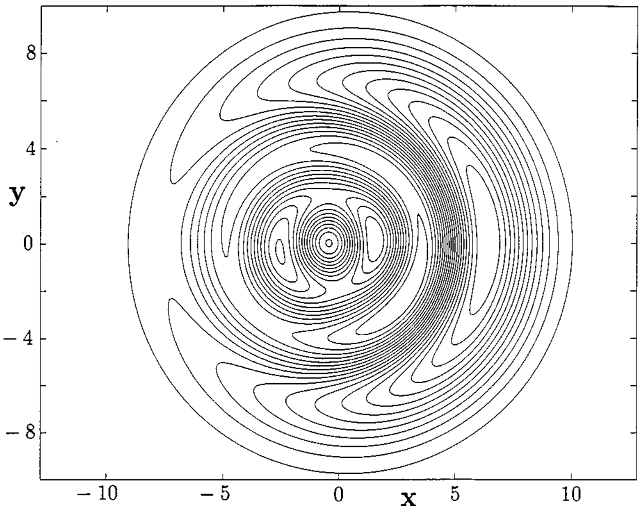

FIG. 8. Asymmetric magnetic surfaces at $z=0$ for plasma equilibrium (5.5).

VI. COUNTEREXAMPLES TO PARKER'S HYPOTHESIS

The general properties of the plasma equilibria were the subject of an interesting discussion in the literature. In his 1979 book [28], Parker writes on p. 374, ''Consider a magnetic field $B_i(x,y) + \epsilon b_i(x,y,z)$ in the neighborhood of the general equilibrium field $B_i(x,y)$,'' and after a detailed study arrives at the conclusion on p. 377, ''Thus, in the general case, we are led to the conclusion that the invariance $\partial b_i/\partial z = 0$ (14.51) is a necessary condition for equilibrium. Any field in which winding pattern changes along the field, so that (14.51) is excluded by the topology, cannot be in equilibrium.''

In [29–31], this conclusion was called ''Parker's theorem.'' Many consequences and generalizations were produced assuming that the theorem is true; see [29–38].

Parker's hypothesis is formulated under three important conditions [28]: (i) ''the local perturbation to the field is small compared to the total field,'' p. 361; (ii) the length of the flux tube L is ''large compared to the characteristic transverse scale of variation l of the field,'' p. 362; and (iii) ''the magnetic field is analytic in its deviation ϵ from the invariant field $B_i(x,y)$,'' p. 378.

Parker's hypothesis attracts considerable attention in the literature. Rosner and Knobloch in [34] study an example of two plasma equilibrium magnetic fields $\mathbf{B}_0(x,y)$ and $\mathbf{B}_1(y,z)$, where the first is z-invariant and the second x-invariant. They treat $\mathbf{B}_1(y,z)$ as a perturbation of $\mathbf{B}_0(x,y)$ and notice that $\mathbf{B}_1(y,z)$ is not z-invariant. However, such a perturbation adds an infinite magnetic energy in any layer $c_1 < z < c_2$ and hence does not satisfy Parker's condition (i). Nor does it satisfy Parker's condition (ii). Moreover, the only exact solutions presented in [34] have singularities: ''$\mathbf{B}_0(x,y) = (x^2+y^2)^{-1}(-y,x,0)$, $\mathbf{B}_1(y,z) = (y^2+z^2)^{-1}(0,-z,y)$, Eq. (3.10).'' Hence, the case treated in [34] is different from the one treated in [28] and therefore it is not a counterexample to Parker's hypothesis.

Van Ballegooijen, in his paper [39], using an expansion parameter different from [28], constructs the force-free perturbations, $p = \text{const}$, of a constant uniform magnetic field B_0 that depend on z. The lowest-order equation [39] is equivalent to the time-dependent two-dimensional vorticity equation and its solutions are supposed to be well-behaved. However, the complete solution in [39] is presented in the form of an infinite power series obtained by subsequent resolving of a more complex system of partial differential equations. Whether this power series is well-behaved in $\mathbb{R}^3$ and whether it satisfies Parker's condition (ii) is not studied. No exact solutions are obtained in [39,40] and the author writes, ''Our conclusions do *not* apply to systems with field lines that are not tied to a boundary. Examples of such systems are the toroidal fields used in fusion machines (e.g. tokamaks),''

[39], p. 426. Hence the work by Van Ballegooijen [39] cannot be considered to supply a counterexample. Therefore, Villata and Tsinganos wrote the following in 1993, [37], p. 2158, in complete contradiction with Van Ballegooijen's results and in complete agreement with Parker's hypothesis: ''It is well known that all well-behaved MHD equilibria extending to all space need to be translationally symmetric.''

The results of the present paper shed some light on this discussion. Indeed, we have derived exact plasma equilibria (3.19) and (5.4) that are smooth and bounded in the whole Euclidean space $\mathbb{R}^3$. The equilibria have no current sheets and no discontinuities and satisfy the above physical conditions (i)–(v). We have constructed a family of global z-invariant plasma equilibria (3.13); each equilibrium possesses a three-dimensional linear space of helically symmetric global perturbations (3.19). The most important feature of these exact solutions is that they *do* depend on the variable z and hence they are *not* translationally invariant, contrary to the predictions in Parker's results. In view of the z quasiperiodicity of the magnetic-field lines, their ''*winding pattern*'' is continuously changing along the variable z and does not repeat.

Let us prove that the plasma equilibria (3.19) and (5.4) satisfy all Parker's conditions [28], pp. 359–391, and hence they form helically symmetric counterexamples to Parker's hypothesis. In [18,19], we present the counterexamples with axial symmetry.

We consider the exact solutions (3.19) and (5.4) as perturbations of the z-invariant global plasma equilibria (3.13). The exact solution (3.13) defines the magnetic field (2.8):

$$\mathbf{B}_N(r)=\frac{\alpha\gamma\psi_N-r\psi_N'}{r^2+\gamma^2}\hat{\mathbf{e}}_z+\frac{\alpha r\psi_N+\gamma\psi_N'}{r^2+\gamma^2}\hat{\mathbf{e}}_\phi,$$

which is nonzero everywhere in the Euclidean space $\mathbb{R}^3$. Indeed, an equality $\mathbf{B}_N(r_0)=0$ implies $\psi_N(r_0)=\psi_N'(r_0)=0$, hence $B_{0N}(x_0)=B_{0N}'(x_0)=0$, a contradiction because all roots of the polynomial $B_{0N}(x)$ are simple; see Appendix A below. Hence for any R and all $r\leqslant R$, we have $|\mathbf{B}_N(r)|\geqslant a_N B(R)>0$.

Let $\mathbf{B}_{Nmn}(r,u)$ be the magnetic field defined by the flux function ψ_{Nmn} (3.19) and $A_N=(|a_{mn}|+|b_{mn}|)/|a_N|$. Inside any domain $0\leqslant r\leqslant R$, we have

$$\frac{|\mathbf{B}_{Nmn}-\mathbf{B}_N|}{|\mathbf{B}_N|}<A_N\frac{C(R)}{B(R)}, \qquad (6.1)$$

where $C(R)$ is a smooth function. At $r\to\infty$, the asymptotics hold,

$$\frac{|\psi_{Nmn}-\psi_N|}{|\psi_N|}<A_N(2\beta)^{n-N}\frac{2C_{Nmn}}{r^{2N-2n-m}},$$

$$C_{Nmn}=\frac{(1-nk_{mn})N!}{(1-Nk_N)n!}. \qquad (6.2)$$

The inequality $2N>2n+m$ implies $|\psi_{Nmn}-\psi_N|/|\psi_N|\to 0$ at $r\to\infty$. Hence for $A_N\ll 1$, we obtain $|\mathbf{B}_{Nmn}-\mathbf{B}_N|/|\mathbf{B}_N|\ll 1$ everywhere in $\mathbb{R}^3$.

Let x_N be the greatest root of the polynomial $B_{0N}(x)$. For $x>x_N$, we have

$$|B_{0N}(x)|>\frac{1-Nk_N}{N!}(x-x_N)^N, \quad |B_{mn}(x)|<\frac{1-nk_{mn}}{n!}x^n.$$

Hence we find

$$\frac{|\psi_{Nmn}-\psi_N|}{|\psi_N|}<A_N C_{Nmn}\frac{r^m x^n}{(x-x_N)^N}.$$

Hence for $x>N^{2\kappa+1}x_N$, $2\kappa=|2\ln C_{Nmn}-m\ln(2\beta)|/(2N-2n-m)\ln N$, we obtain

$$\frac{|\psi_{Nmn}-\psi_N|}{|\psi_N|}<A_N.$$

The same inequality is true for the magnetic field. Thus for $A_N\ll 1$, the perturbations (3.19) can be significant only for $x<N^{2\kappa+1}x_N$. Substituting $x=2\beta l^2$, we find for the length scale l in the variable r of the perturbations (3.19)

$$l\leqslant N^\kappa\left(\frac{Nx_N}{2\beta}\right)^{1/2}.$$

The inequalities (6.1) and (6.2) mean that the plasma equilibria (3.19) at $A_N\ll 1$ are small perturbations in the whole Euclidean space $\mathbb{R}^3$ of the z-invariant equilibrium (3.13). Hence we obtain that Parker's condition which states that ''the local perturbation to the field is small compared to the total field,'' [28], p. 361 is satisfied everywhere. Parker's condition that ''the magnetic field is analytic in its deviation ϵ from the invariant field $B_i(x,y)$,'' [28], p. 378, is satisfied because the exact solutions (3.19) are linear functions of small parameters a_{mn}, b_{mn}. Parker's condition that the length of the flux tube L is ''large compared to the characteristic transverse scale of variation l of the field,'' [28], p. 362, is satisfied because $l\leqslant N^\kappa\sqrt{Nx_N/2\beta}$ and the flux tube length L can be taken arbitrarily large for the z-invariant equilibrium (3.19). Hence $L\gg l$. All perturbations (3.19) *are not* z-invariant.

One of the origins of the discrepancy with Parker's results is as follows. In his book [28], Parker writes (p. 369), ''We suppose for convenience that, although $B_z(x,y)$ may vary widely, it does not vanish and change sign,'' and he arrives at the statement, ''The result can be written

$$\frac{\partial}{\partial x}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial x}+\frac{\partial}{\partial y}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial y}+\frac{\partial}{\partial z}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial z}=0. \qquad (6.3)$$

This form is totally elliptic. In an infinite space its only bounded solutions are constants, $\Psi=C$.''

This statement is used as a key argument in the proof of Parker's theorem on pp. 369 and 370, [28] and also in the proof of its generalization for magnetohydrodynamics [30], p. 837, Eq. (62).

We show that the statement is a logical error. Indeed, let us consider one concrete example:

$$B_z(x,y)=[1+(ax+by)^2]^{-1/2}, \quad \Psi(x,y)=\tan^{-1}(ax+by), \qquad (6.4)$$

where $\tan^{-1}(z)$ is the inverse function for $\tan(z)$ and $a,b=\text{const}$. Function $B_z(x,y)$ (6.4) *does* satisfy Parker's condition because it is nonvanishing throughout the entire space,

$-\infty < x,y < +\infty$. Function $\Psi(x,y)$ (6.4) satisfies Eq. (6.3). It is bounded, $|\Psi(x,y)| < \pi/2$, and it is *nonconstant*.

Remark 4. Solutions (6.4) can be generalized in different ways. For example, for any harmonic function $h(x,y)$, $\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2 = 0$, the functions

$$B_z(x,y) = [1 + h^2(x,y)]^{-1/2}, \quad \Psi(x,y) = \tan^{-1}(h(x,y)) \tag{6.5}$$

satisfy Eq. (6.3). It is evident that function $B_z(x,y)$ (6.5) is nonvanishing for all x,y and function $\Psi(x,y)$ (6.5) is bounded and *nonconstant*.

VII. SUMMARY

We have developed a model of astrophysical jets outside of their accretion disks. The model satisfies the physical conditions (i)–(v) of Sec. II and is represented by the exact helically symmetric solutions to the MHD equilibrium equations (1.3) and (1.4). The modeled astrophysical jets are highly collimated due to the rapid decreasing of the magnetic field in the transversal direction, $|\mathbf{B}| \approx c_N \exp(-\beta r^2) r^{2N}$ at $r \to \infty$. The magnetic-field lines form a combination of nested and helically rotated cylindrical magnetic surfaces.

The constructed MHD equilibria are based on the exact solutions (3.19) and (5.4) for the flux functions $\psi(r,z - \gamma\phi)$, which define the magnetic field $\mathbf{B}_1$. The equilibria depend upon two arbitrary functions $\alpha(\psi)$ and the plasma density $\rho(\psi) \geqslant 0$:

$$\mathbf{B} = k \,\mathrm{ch}\, \alpha(\psi)\mathbf{B}_1, \quad \mathbf{V} = \frac{k \,\mathrm{sh}\, \alpha(\psi)}{\sqrt{\mu\rho(\psi)}}\mathbf{B}_1.$$

The second family of equilibria has the form

$$\mathbf{B} = k \,\mathrm{sh}\, \alpha(\psi)\mathbf{B}_1, \quad \mathbf{V} = \frac{k \,\mathrm{ch}\, \alpha(\psi)}{\sqrt{\mu\rho(\psi)}}\mathbf{B}_1.$$

For these exact solutions, the ratio of the plasma magnetic and kinetic energy $\mathbf{B}^2/(\mu\rho\mathbf{V}^2)$ is variable in the space $\mathbb{R}^3$ and is constant on the magnetic surfaces $\psi(x) = \mathrm{const}$. Hence the derived MHD equilibria generalize the classical Chandrasekhar equipartition solution [20].

For the equilibria (3.19) and (5.4), the generic magnetic-field lines are quasiperiodic in z, which implies that the magnetic lines never repeat in the z direction, but can have a structure arbitrarily close to the initial data. Their winding pattern changes continuously with z, and does not repeat.

Up until now, the quote ''It is well known that all well-behaved MHD equilibria extending to all space need to be translationally symmetric,'' [37], p. 2158, has been generally accepted. We have proved that this is logically incorrect because even small perturbations (3.19) and (5.4) of the translationally symmetric plasma equilibria (3.13) have no translational symmetry. The exact plasma equilibria (3.19) and (5.4) provide helically symmetric counterexamples to Parker's hypothesis [28]. In [18,19], we present the axially symmetric counterexamples.

The obtained results shed some light also on the ongoing discussion in the MHD literature about ''*unavoidable*'' and ''*ubiquitous*'' singularities (current sheets), which are supposed to arise when a magnetic flux tube is perturbed [41]. The derived exact global plasma equilibria (3.19) and (5.4) are smooth everywhere and have no tangential discontinuities and no current sheets for any values of their arbitrary functions $\alpha(\psi)$ and $\rho(\psi)$ and arbitrary parameters. The axially symmetric plasma equilibria with the same properties are derived in [18,19]. The exact solutions (5.4) depend on $2K+2$ arbitrary parameters $\gamma, a_N, a_{mn}, b_{mn}$ and have the form of the Fourier series in variable $u = z - \gamma\phi$, for a generic $r = r_0$. These solutions approximate any smooth function $\psi(r_0,u)$ that is $(2\pi\gamma)$-periodic in u. Therefore, the exact solutions (5.4) describe rather generic global helically symmetric MHD equilibria.

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APPENDIX A

To obtain the explicit form of the polynomials $B_{mn}(x)$, we make the substitution

$$B(x) = P(x) - k_{mn} x P'(x) \tag{A1}$$

to Eq. (3.6). A direct verification proves the identity

$$\begin{aligned}(x^2 + c_1 x)B'' &+ [-x^2 + (m - c_1)x + (m+1)c_1] \\ &\times B' + n(x + c_1 - k_{mn}c_1)B \\ &= (x + c_1 - k_{mn}c_1)Q - k_{mn}x(x + c_1) \\ &\times Q' + (nc_1k_{mn}^2 - (m + c_1)k_{mn} - 1)xP',\end{aligned}$$

where $Q = xP'' + (1 + m - x)P' + nP$. The necessary condition (3.5) is equivalent to the equation

$$nc_1k_{mn}^2 - (m + c_1)k_{mn} - 1 = 0, \quad k_{mn} = \frac{m + c_1 - \alpha\gamma}{2nc_1}.$$

Equation $Q = 0$ or

$$xP'' + (1 + m - x)P' + nP = 0$$

has polynomial solutions $P_{mn}(x) = d^m L_{m+n}(x)/dx^m$ of degree n, where $L_p(x)$ are the Laguerre polynomials (3.8). Hence using formula (A1), we obtain that if the two integers m and n satisfy Eq. (3.5), then Eq. (3.6) follows from $Q = 0$ and has the polynomial solution (3.7).

Using formulas (3.8), we find the first three polynomials (3.14):

$$B_{01}(x) = 1 - (1 - k_1)x,$$

$$B_{02}(x) = 1 - 2(1 - k_2)x + \tfrac{1}{2}(1 - 2k_2)x^2, \tag{A2}$$

$$B_{03}(x) = 1 - 3(1 - k_3)x + \tfrac{3}{2}(1 - 2k_3)x^2 - \tfrac{1}{6}(1 - 3k_3)x^3.$$

The first three polynomials $B_{1n}(x)$ (3.7) have the form

$$B_{10}(x)=-1,\quad B_{11}(x)=-2+(1-k_{11})x,$$
$$B_{12}(x)=-3+3(1-k_{12})x-\tfrac{1}{2}(1-2k_{12})x^2. \tag{A3}$$

Equation (3.5) implies $m+c_1<\alpha\gamma$, hence $k_{mn}<0$. Hence using the classical properties of the Laguerre polynomials (3.9), we obtain that the new polynomials $B_{mn}(x)$ (3.7) have alternating coefficients and each polynomial $B_{mn}(x)$ has n distinct positive roots for any $m\geqslant 0$.

APPENDIX B

Let us calculate the rotation transform T_h: $S_0\to S_0=S_1$ for the thin elliptical cylinders C_h rotating around their magnetic axes. Let an exact solution (3.19) have its local maximum (or minimum) at a point (r_0,u_0); $\psi(r_0,u_0)=\psi_0$. The critical point (r_0,u_0) defines the magnetic axis M_0: $r=r_0$, $z-\gamma\phi=u_0$. We have an approximation

$$\psi(r_0+\delta r,u_0+\delta u)\approx\psi_0+a(\delta r)^2+b(\delta u)^2+2c\,\delta r\,\delta u,$$

where $2a=\psi_{rr}(r_0,u_0)$, $2b=\psi_{uu}(r_0,u_0)$, $2c=\psi_{ru}(r_0,u_0)$. For points (r_0,u_0) of maximum or minimum, we have $\psi_{rr}\psi_{uu}-\psi_{ru}^2\geqslant 0$. In a small neighborhood of the helix M_0, the magnetic surfaces $\psi(r,u)=\text{const}$ are thin elliptical cylinders C_h rotating around M_0.

Let us find the rotation number θ_0 for the curves S_0 at $h\ll 1$. The curve S_0 is approximated by the ellipse $a(\delta r)^2+b(\delta u)^2+2c\,\delta r\,\delta u=h^2$ with parametrization

$$\delta r=\kappa h\sin(2\pi s),$$

$$\delta u=-c\kappa h\sin(2\pi s)/b+h\cos(2\pi s)/\sqrt{b},$$

where $\kappa=\sqrt{b}/\sqrt{ab-c^2}$ and $\delta u=-\gamma\delta\phi$. The magnetic field (6) defines equations for the magnetic lines:

$$\dot r=\frac{\psi_u}{r},\quad \dot z=\frac{\alpha\gamma\psi-r\psi_r}{r^2+\gamma^2},\quad \dot\phi=\frac{\alpha r\psi+\gamma\psi_r}{r(r^2+\gamma^2)}. \tag{B1}$$

On the magnetic axis M_0, we have $z(t)=\alpha\gamma\psi_0 t/(r_0^2+\gamma^2)+z_0$. Hence the shift of z for the period $2\pi\gamma$ takes place at $t_0=2\pi(r_0^2+\gamma^2)/(\alpha\psi_0)$. Equations (B1) imply the equations for the variations,

$$r_0\dot{\delta r}=2b\,\delta u+2c\,\delta r,\quad r_0\dot{\delta u}=-2a\,\delta r-2c\,\delta u.$$

Their solutions are

$$\delta r(t)=\kappa h\sin(\omega t+2\pi s),$$

$$\delta u(t)=-c\kappa h\sin(\omega t+2\pi s)/b+h\cos(\omega t+2\pi s)/\sqrt{b},$$

where $\omega=2\sqrt{ab-c^2}/r_0$. For $t=t_0$, we obtain the rotation transform $T(s)=s+t_0\sqrt{ab-c^2}/(\pi r_0)$. Hence the rotation number is

$$\theta_0=\frac{(r_0^2+\gamma^2)\sqrt{\psi_{rr}\psi_{uu}-\psi_{ru}^2}}{\alpha r_0\psi_0}. \tag{B2}$$

In general, the rotation number θ_0 (B2) is irrational and depends on the small parameter h. Hence the generic magnetic-field lines are quasiperiodic in variables r, ϕ and in variable $z \bmod 2\pi\gamma$.

APPENDIX C

The third equation of Eqs. (1.4) implies the existence of magnetic surfaces for the generic MHD equilibria. Indeed, equation $\text{curl}(\mathbf{V}\times\mathbf{B})=0$ in any simply connected domain D yields $\mathbf{V}\times\mathbf{B}=\text{grad}\,\psi(x)$, where $\psi(x)$ is some smooth function in D. The surfaces $\psi(x)=\text{const}$ are magnetic surfaces because $(\mathbf{B}\cdot\text{grad}\,\psi)=0$, $(\mathbf{V}\cdot\text{grad}\,\psi)=0$.

Formulas (1.6) are manifestations of the following new symmetries of the magnetohydrodynamics equilibrium equations (1.3) and (1.4). Let $\mathbf{B}(x)$, $\mathbf{V}(x)$, $P(x)$, $\rho(x)$ be an arbitrary solution for which $\rho=\rho[\psi(x)]$, for example, $\rho=\text{const}$. Here $\psi(x)$ can be any function that defines magnetic surfaces. The symmetries transform the solution $\mathbf{B}$, $\mathbf{V}$, P, ρ into a continuous family of new solutions,

$$\mathbf{B}_1=a\mathbf{B}+b\sqrt{\mu\rho}\mathbf{V},\quad \mathbf{V}_1=\frac{b}{\sqrt{\mu\rho_1}}\mathbf{B}+a\sqrt{\frac{\rho}{\rho_1}}\mathbf{V}, \tag{C1}$$

$$P_1=C_1P-\frac{b^2}{2\mu}\mathbf{B}^2-\frac{1}{2}b^2\rho\mathbf{V}^2-ab\sqrt{\frac{\rho}{\mu}}(\mathbf{B}\cdot\mathbf{V}), \tag{C2}$$

where functions $a(\psi)$ and $b(\psi)$ are constant on magnetic surfaces and satisfy the equation $a^2(\psi)-b^2(\psi)=C_1=\text{const}$, and the plasma density $\rho_1(\psi)\geqslant 0$ is arbitrary. The fact that the functions $\mathbf{B}_1(x)$, $\mathbf{V}_1(x)$, $P_1(x)$, $\rho_1(x)$ satisfy Eqs. (1.3) and (1.4) is proved by a direct substitution and using identity (2.1). The symmetries (C1) and (C2) have the following physical meaning. The difference between the plasma kinetic and magnetic energies is changed by a scalar C_1 multiplication. In addition, $\sqrt{\rho_1}\mathbf{V}_1\times\mathbf{B}_1=C_1\sqrt{\rho}\mathbf{V}\times\mathbf{B}$.

For $\rho_1(\psi)=\rho(\psi)$, the nondegenerate transformations

$$\mathbf{B}_1=a\mathbf{B}+b\sqrt{\mu\rho}\mathbf{V},\quad \mathbf{V}_1=\frac{b}{\sqrt{\mu\rho}}\mathbf{B}+a\mathbf{V} \tag{C3}$$

form an infinite-dimensional Lie group G. The group G has two components: G_1: $a^2(\psi)-b^2(\psi)=k^2$, and G_2: $a^2(\psi)-b^2(\psi)=-l^2$, where $k\neq 0$ and $l\neq 0$. These equations are resolved by the formulas $a(\psi)=k\,\text{ch}\,\alpha(\psi)$, $b(\psi)=k\,\text{sh}\,\alpha(\psi)$ for G_1, and $a(\psi)=l\,\text{sh}\,\beta(\psi)$, $b(\psi)=l\,\text{ch}\,\beta(\psi)$ for G_2, where $\alpha(\psi)$ and $\beta(\psi)$ are arbitrary functions of ψ, and k, l are arbitrary nonzero reals. Hence elements of G_1 and G_2 are parametrized by the pairs $[k,\alpha(\psi)]$ and $[l,\beta(\psi)]$. We have $G_1\cdot G_1\subset G_1$, $G_1\cdot G_2\subset G_2$, $G_2\cdot G_2\subset G_1$. The composition of transformations (C3) implies the Abelian law of the group multiplication $[k_1,\alpha_1(\psi)][k_2,\alpha_2(\psi)]=[k_1k_2,\alpha_1(\psi)+\alpha_2(\psi)]\in G$, which completely defines the Lie group G. Applying the symmetries (C1) and (C2) to any exact solution of the plasma equilibrium equations (1.2), we obtain exact MHD equilibria (1.6).

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